A GAMMA IBNR CLAIMS RESERVING MODEL WITH DEPENDENT DEVELOPMENT PERIODS

Topic 3: Liability Risk – Reserve Models

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Abstract

Two distribution dependent IBNR claims reserving models with gamma distributed paid claims are considered. The first model assumes independent development periods and allocates the coefficient of variation of the total ultimate claims of a line of business with multiple underwriting periods to the coefficient of variation of the total ultimate claims of a single underwriting period inverse proportionally to the squared-root premium volumes. The second model extension is based on a simple Fréchet like multivariate distribution, which models the whole range of dependence between independence and comonotone dependence. The chosen model uses only one additional dependence parameter, which is chosen such that it yields the most conservative model for IBNR claims reserving with respect to the concordance order for the bivariate margins of this model. The use of the introduced models is compared with the results obtained through application of a recent optimal credible loss ratio IBNR reserving method.

Key words

IBNR reserves, dependence, coefficient of variation, gamma distribution

1. Introduction.

A main and very important advantage of the classical IBNR claims reserving methods, like the chain-ladder, Cape Cod and Bornhuetter-Ferguson methods or credibility like methods (e.g. Mack(2000), Hürlimann(2005)), is their distribution-free validity. However, in a risk consulting environment, there is an accrued interest to know more about the standard deviation and the higher percentile values. Therefore, attempts to model adequately not only the mean of the IBNR claims reserves but also its full distribution have the potential to retain more attention from both a theoretical and practical viewpoint. Early developments in this area include work by Bühlmann et al.(1980), and Hertig(1985). The present approach is inspired from Mack(1997), which proposes distribution dependent IBNR claims reserving methods, in particular a cross-classified parametric method of multiplicative type based on the gamma distribution. A more detailed account of the content of the present paper follows.

In Section 2, we consider a first IBNR claims reserving model (I) with independently distributed gamma paid claims, which is based on Mack(1997), Section 3.3.3, p. 281-283, but slightly modified in view of a recent "homogeneous allocation principle" motivated and considered first in Hürlimann(2002). Under suitable assumptions, this principle allocates the coefficient of variation of the total ultimate claims of a line of business with multiple underwriting periods to the coefficients of variations of the total ultimate claims of the single underwriting periods. This allocation principle states that the single coefficients of variation are inverse proportional to the squared-root premium volumes. The $2n+1$ parameters of the IBNR model are estimated using the method of maximum likelihood estimation. Formulas for the means and variances of the IBNR claims reserves are given.

In Section 3, the above model is extended to a second gamma IBNR claims reserving model (II) with dependent development periods. The extension is based on a simple Fréchet like multivariate distribution, which models the whole range of dependence between independence and comonotone dependence. The chosen model uses only one additional dependence parameter, which is chosen such that it yields the most conservative model for IBNR claims reserving with respect to the concordance order for the bivariate margins of this model.

Section 4 illustrates the use of the introduced models and compare them with the results obtained through application of the recent optimal credible loss ratio IBNR reserving method considered in Hürlimann(2005).

2. A gamma model with independent development periods.

The *total ultimate claims* of the claims incurred in a given year, known in the future when all claims have been closed and paid out, is defined as follows:

total ultimate claims = paid claims + outstanding claims reserve

+ IBNR claims reserve (**I**ncurred **B**ut **N**ot **R**eported claims reserve)

Let n be the number of periods (to fix ideas one-year periods) for which historical data on paid claims is available. Let S_{ik}, $1 \le i,k \le n$, be the claims occurred in period i and reported in period $i+k-1$. The restriction of claims to n development periods yields the condition $i+k-1 \le n$. Under the assumption that after n development periods all claims that occurred in a period are known and closed, the amount $\sum_{k=1}^{n} S_{ik}$ is the total amount of claims occurred in period i. However, at the end of development period $m-i+1$ only the amount $\sum_{k=1}^{n-i+1} S_{ik}$ is known. Therefore the required amount for the incurred but not reported claims for period i at the end of development period $m-i+1$ (called the *i-th period IBNR claims reserve*) is equal to $R_i = \sum_{k=n-i+2}^{n} S_{ik}$, $i = 2,...,n$. An estimate for the total required amount of incurred but not reported claims over all periods $R = \sum_{i=2}^{n} R_i$ is called *total IBNR reserve*. In the following, the sums $C_{ik} = \sum_{j=1}^{k} S_{ij}$, $1 \le i,k \le n$, denote the accumulated paid claims occurred in period i and reported in period $i+k-1$.

Usually, the estimation of IBNR claims reserves is based on a full loss triangle of paid claims statistics S_{ik}, $1 \le i,k \le n$, subject to the restriction $i+k-1$. Some methods require additionally the knowledge of a measure of exposure V_i for each underwriting period $i = 1,...,n$. To fix ideas, we suppose that V_i are actuarial premiums.

Remark 2.1.

It is possible to use other exposure measures. For example, suppose that instead of actuarial premiums the number N_i of accumulated claims for the underwriting period i at the latest period of development n is known. Then one proceeds as follows to estimate an exposure measure V_i similar to an actuarial premium measure. Consider the total burning costs of the latest period of development $BC = \sum_{i=1}^{n} C_{in-i+1}$ and let $N = \sum_{i=1}^{n} N_i$ be the total number of claims for the same period of development. Then the ratio BC/N represents the burning cost per claims and $V_i = (N_i/N) \cdot BC$ is an appropriate exposure measure, which can be used instead of actuarial premiums.

An essential part of the proposed distribution dependent IBNR claims reserving models rely on the following "homogenous allocation principle" introduced in Hürlimann(2002), which is carried forward to the present context.

Consider the total ultimate claims of a line of business for n underwriting periods, which are divided into the total ultimate claims of each underwriting period. We are interested in the following quantities:

V	:	premium volume of the line of business
U	:	total ultimate claims of the line of business
$\mu = E[U]$	:	mean of the total ultimate claims of the line of business
$k = CoV[U]$	:	coefficient of variation of the line of business
V_i	:	premium volume of the i-th underwriting period
U_i	:	total ultimate claims of the i-th underwriting period

$\mu_i = E[U_i]$: mean total ultimate claims of the i-th underwriting period
$k_i = CoV[U_i]$: coefficient of variation total ultimate claims of the i-th underwriting period

Applying the *collective model of risk theory*, one approximates the total ultimate claims random variables U_i by *compound Poisson* random variables

$$U_i = \sum_{j=1}^{N_i} Y_{i,j}, \quad i = 1,...,n, \tag{2.1}$$

where N_i represents the *number of claims* and $Y_{i,j}$ represents the *claim size* given a claim has occurred. The model supposes that N_i is Poisson(λ_i) distributed, the $Y_{i,j}$'s are independent and identically distributed and they are independent from N_i (e.g. Beard et al.(1984), Section 3). Denote the identical claim sizes by $Y_i = Y_{i,j}, \quad j = 1,..., N_i$. Furthermore, assume that the total ultimate claims of the underwriting periods are independent random variables. Then the total ultimate claims random variable U of the line of business is again compound Poisson such that

$$U = \sum_{\ell=1}^{N} Z_\ell, \tag{2.2}$$

where N is Poisson(λ) distributed, with $\lambda = \sum_{i=1}^{n} \lambda_i$, the Z_ℓ's are independent and identically distributed, with $Z_\ell = \sum_{i=1}^{n} \frac{\lambda_i}{\lambda} \cdot Y_i$, and the Z_ℓ's are independent from N. The identical claim sizes are denoted by $Z = Z_\ell, \quad \ell = 1,..., N$. Under a simple *homogeneity assumption*, the main characteristics of the underwriting periods relate to those of the whole line of business as follows.

Theorem 2.1. (Hürlimann(2002)) In the above collective model of risk theory for the total ultimate claims, suppose that the claim sizes of the underwriting periods are identically distributed, that is $Y_i = Y$ for all $i = 1,..., n$. Assume the premium volumes are calculated according to the expected value principle such that $V = (1+\theta) \cdot \mu$, $V_i = (1+\theta) \cdot \mu_i, i = 1,..., n$, with θ the loading factor. Then the means and coefficients of variation $(\mu_i, k_i), i = 1,..., n$, relate to (μ, k) according to the following rules:

(R1) $$\frac{\mu_i}{V_i} = \frac{\mu}{V}, \quad i = 1,..., n$$

(invariance of the mean total ultimate claims per unit of premium volume)

(R2) $$k_i = \sqrt{\frac{V}{V_i}} \cdot k, \quad i = 1,..., n$$

(coefficients of variation inverse proportional to the square-root premium volumes)

After these preliminaries, let us consider the following IBNR claims reserving model with independently distributed gamma paid claims, which is inspired from Mack(1997), p. 281-83, but slightly modified in view of the rule (R2) of Theorem 2.1.

Gamma IBNR claims reserve model (I)

Under the assumptions of Theorem 2.1 on the total ultimate claims, assume that the paid claims S_{ik}, $1 \le i,k \le n$, satisfy the following model assumptions:

(I.1) $E[S_{ik}] = \left(\frac{V_i}{V}\right) \cdot x_i y_k$, for some parameters x_i, y_k, $1 \le i,k \le n$,

(I.2) $CoV[S_{ik}] = CoV[U_i] = \sqrt{\frac{V}{V_i}} \cdot CoV[U] = \sqrt{\frac{V}{V_i}} \cdot \sqrt{\frac{1}{\alpha}}$, for some parameter $\alpha > 0$,

(I.3) The paid claims S_{ik}, $1 \le i,k \le n$, follow independent gamma distributed random variables

Under these model assumptions, the probability density of the paid claims S_{ik} is of the following form:

$$f_{ik}(s) = \Gamma(\alpha_i)^{-1} \left(\frac{s}{\beta_{ik}}\right)^{\alpha_i} \cdot \frac{\exp\left(-\frac{s}{\beta_{ik}}\right)}{s}, \quad 1 \le i,k \le n,$$
$$\alpha_i = \left(\frac{V_i}{V}\right) \cdot \alpha, \quad \beta_{ik} = \frac{1}{\alpha} \cdot x_i y_k. \tag{2.3}$$

In practice, the loss triangle of paid claims $S_{ik}, 1 \le i \le n, k \le n-i+1$, as well as the premium volumes, will be given, and the parameters x_i, y_k, $1 \le i,k \le n$, and α of the model have to be estimated. The method of maximum likelihood estimation is used. Under the assumption that the observed paid claims are independent, the likelihood function equals

$$L = \prod_{i=1}^{n} \prod_{k=1}^{n-i+1} \exp\left(-\frac{\alpha S_{ik}}{x_i y_k}\right) \cdot \left(\frac{\alpha S_{ik}}{x_i y_k}\right)^{\alpha \frac{V_i}{V}} \cdot \frac{1}{S_{ik} \Gamma\left(\alpha \frac{V_i}{V}\right)}. \tag{2.4}$$

Therefore, the log-likelihood function is given by

$$\ln(L) = \sum_{i=1}^{n} \sum_{k=1}^{n-i+1} \left\{ -\frac{\alpha S_{ik}}{x_i y_k} + \alpha \frac{V_i}{V} \cdot \ln\left(\frac{\alpha S_{ik}}{x_i y_k}\right) - \ln\left(S_{ik} \Gamma\left(\alpha \frac{V_i}{V}\right)\right) \right\}. \tag{2.5}$$

The maximum likelihood estimators of the parameters x_i, y_k, $1 \le i,k \le n$, and α, are those values, which maximize L respectively $\ln(L)$, that is the solutions of the equations

$$0 = \partial \ln(L) / \partial x_i = \alpha \cdot \sum_{k=1}^{n-i+1} \left(\frac{S_{ik}}{x_i^2 y_k} - \frac{V_i}{V x_i} \right), \quad i = 1,...,n, \tag{2.6}$$

$$0 = \partial \ln(L) / \partial y_k = \alpha \cdot \sum_{i=1}^{n-k+1} \left(\frac{S_{ik}}{x_i y_k^2} - \frac{V_i}{V y_k} \right), \quad k = 1,...,n, \tag{2.7}$$

$$0 = \partial \ln(L) / \partial \alpha = \sum_{i=1}^{n} \sum_{k=1}^{n-i+1} \frac{V_i}{V} \left\{ \ln\left(\frac{\alpha S_{ik}}{x_i y_k} \right) - \psi\left(\alpha \frac{V_i}{V} \right) \right\}, \tag{2.8}$$

where $\psi(x) = \Gamma'(x) / \Gamma(x)$ is the digamma function. Now, the maximum likelihood estimates $\hat{x}_i, \hat{y}_k$, $1 \le i,k \le n$, are already determined by the equations (2.6), (2.7) and given by

$$\hat{x}_i = \frac{V}{(n-i+1)V_i} \sum_{k=1}^{n-i+1} \frac{S_{ik}}{y_k}, \quad i = 1,...,n, \quad \hat{y}_k = \frac{V}{\sum_{i=1}^{n-k+1} V_i} \sum_{i=1}^{n-k+1} \frac{S_{ik}}{x_i}, \quad k = 1,...,n. \tag{2.9}$$

This system of equations can be solved iteratively through alternate calculation of $\hat{x}_i$ and $\hat{y}_k$ starting with the values $y_1 = ... = y_n = 1$. Inserted into (2.8) this yields the maximum likelihood estimate $\hat{\alpha}$ as implicit solution of the equation

$$\sum_{i=1}^{n} \frac{V_i}{V} \cdot \left\{ (n-i+1) \cdot \left[\ln(\alpha) - \psi\left(\alpha \frac{V_i}{V} \right) \right] + \sum_{k=1}^{n-i+1} \ln\left(\frac{S_{ik}}{\hat{x}_i \hat{y}_k} \right) \right\} = 0. \tag{2.10}$$

It is now possible to estimate in a simple way the mean and variance of the IBNR claims reserves through the formulas

$$\hat{R}_i = \sum_{k=n-i+2}^{n} \hat{E}[S_{ik}] = \left(\frac{V_i}{V} \right) \cdot \hat{x}_i \cdot \sum_{k=n-i+2}^{n} \hat{y}_k, \quad i = 2,...,n, \tag{2.11}$$

$$\widehat{Var}[R_i] = \sum_{k=n-i+2}^{n} \widehat{Var}[S_{ik}] = \left(\frac{V_i}{V} \right) \cdot \frac{1}{\alpha} \cdot \hat{x}_i^2 \cdot \sum_{k=n-i+2}^{n} \hat{y}_k^2, \quad i = 2,...,n. \tag{2.12}$$

Even more, by the assumption (I.3) of independence, the full distribution of the IBNR claims reserve is in principle known. Indeed, the exact distribution of independent gamma distributed random variables has been studied many times in the statistical literature. Johnson et al.(1994), pp. 384-85, refers to Mathai(1982), Moschopoulos(1985) and Sim(1992). One can add Provost(1989), which determines the exact density applying the inverse Mellin transform and Hürlimann(2001), who obtains new analytical expressions, which are similar but different to previous expressions by Provost(1989) and Sim(1992). However, the obtained expressions are not suitable for a straightforward evaluation of the percentile values of the distribution. Fortunately, the Example 4.2 in Hürlimann(2002) shows that the exact distribution can for practical purposes be approximated by a gamma distribution with the exact estimated mean and variance as given in (2.11), (2.12). In the present paper, we will therefore assume that the full distribution of the IBNR claims reserve $R_i, i = 2,...,n,$ for the Gamma model (I) has distribution $\Gamma(\beta_i x, \alpha_i)$, $i = 2,...,n,$ where

$\Gamma(x;\alpha)=\Gamma(\alpha)^{-1}\cdot\int_0^x t^{\alpha-1}e^{-t}dt$ is the incomplete gamma function, and the parameters are given by

$$\alpha_i=\frac{1}{k_i^2},\quad \beta_i=\frac{1}{k_i^2\mu_i},$$
$$\mu_i=\frac{V_i}{V}x_i\cdot\sum_{k=n-i+2}^{n}y_k,\quad k_i^2=\frac{\frac{V_i}{V\alpha}x_i^2\cdot\sum_{k=n-i+2}^{n}y_k^2}{\mu_i^2}. \tag{2.13}$$

3. Extension to a gamma model with dependent development periods.

To keep a multivariate generalization of the Gamma IBNR claims reserve model (I) as simple as possible, we assume that the bivariate margins of the paid claims random vector $S^{(i)}=(S_{i1},S_{i2},...,S_{in-i+2})$ belong to the same parametric family of linear Spearman copulas such that

$$F_{i,k,\ell}(x,y)=(1-\theta_{k\ell})\cdot F_{i,k,\ell}^{\perp}(x,y)+\theta_{k\ell}\cdot F_{i,k,\ell}^{+}(x,y),\quad 0\le\theta_{k\ell}\le 1, \tag{3.1}$$

where $\theta_{k\ell}$ identifies with Spearman's grade correlation coefficient, $F_{i,k,\ell}^{\perp}(x,y)$ is the bivariate distribution of an independent version $(S_{ik}^{\perp},S_{i\ell}^{\perp})$ of $(S_{ik},S_{i\ell})$, and $F_{i,k,\ell}^{+}(x,y)$ is the distribution of a comonotone version $(S_{ik}^{+},S_{i\ell}^{+})$ of $(S_{ik},S_{i\ell})$. For simplicity, we restrict ourselves here with the situation of positive dependent margins, though negatively dependent margins could be included without further difficulty. Based on mixtures of independent conditional distributions, and bivariate margins (3.1), it is possible to construct multivariate distributions, which fulfil the desirable properties postulated in Joe(1997), Section 4.1 (e.g. Hürlimann(2004)).

Though presumably possible, the evaluation of the full distribution of IBNR reserves using these multivariate distributions should be rather technical. Instead of this, we consider the simple Fréchet like multivariate distribution

$$F_i^{*}(x)=(1-\theta)\cdot F_i^{\perp}(x)+\theta\cdot F_i^{+}(x),\quad 0\le\theta\le 1,\quad x=(x_1,...,x_{n-i+2}), \tag{3.2}$$

where $F_i^{\perp}(x)$ is the multivariate distribution of an independent version of $S^{(i)}$ and $F_i^{+}(x)$ is the multivariate distribution of a comonotone version of $S^{(i)}$. Intuitively, the distribution (3.2) models the whole range of possible dependence structure between independence and comonotone dependence. It would also be possible to model similarly the possible dependence structure between independence and "minimal" dependence.

The practical application of this model is motivated as follows. Guided by the concern of "prudent" IBNR claims reserving, we require that the bivariate margins of (3.2) are at least as positively dependent as those obtained from the bivariate model (3.1) in the concordance ordering (e.g. Cambanis et al.(1976), Tchen(1980), Dhaene and Goovaerts(1996)). Consider the bivariate distribution of the bivariate margin of (3.2), which are given by

$$F_{i,k,\ell}^{*}(x,y)=(1-\theta)\cdot F_{i,k,\ell}^{\perp}(x,y)+\theta\cdot F_{i,k,\ell}^{+}(x,y). \tag{3.3}$$

Then the stated condition means that $F_{i,k,\ell}(x,y) \le F^{*}_{i,k,\ell}(x,y)$, and this is equivalent to

$$(\theta - \theta_{k\ell}) \cdot F^{\perp}_{i,k,\ell}(x,y) \le (\theta - \theta_{k\ell}) \cdot F^{+}_{i,k,\ell}(x,y) . \tag{3.4}$$

Since $F^{\perp}_{i,k,\ell}(x,y) \le F^{+}_{i,k,\ell}(x,y)$ this means that

$$\theta \ge \theta_{k\ell}, \quad \text{for all } (k,\ell). \tag{3.5}$$

Therefore, for prudent IBNR evaluation, it is reasonable to set the dependence parameter of the model (3.2) equal to

$$\theta = \max_{(k,\ell)} \{\theta_{k\ell}\}. \tag{3.6}$$

This model choice yields the most conservative model for IBNR claims reserving with respect to the concordance order for the bivariate margins of this model.

In view of the above discussion, it makes sense to consider the following extension of Model (I) to the situation of dependent development years.

Gamma IBNR claims reserve model (II)

Under the assumptions of Theorem 2.1 on the total ultimate claims, assume that the paid claims S_{ik}, $1 \le i,k \le n$, satisfy the following model assumptions:

(II.1) $E[S_{ik}] = \left(\frac{V_i}{V}\right) \cdot x_i y_k$, for some parameters x_i, y_k, $1 \le i,k \le n$,

(II.2) $CoV[S_{ik}] = CoV[U_i] = \sqrt{\frac{V}{V_i}} \cdot CoV[U] = \sqrt{\frac{V}{V_i}} \cdot \sqrt{\frac{1}{\alpha}}$, for some parameter $\alpha > 0$,

(II.3) The random vector of paid claims $S^{(i)} = (S_{i1}, S_{i2}, ..., S_{in-i+2})$, $i = 2,...,n$, follows a Fréchet like multivariate distribution of the type $F^{*}_{i}(x) = (1-\theta) \cdot F^{\perp}_{i}(x) + \theta \cdot F^{+}_{i}(x)$ with gamma distributed margins

To evaluate the mean, standard deviation and percentiles of the IBNR claims reserves, the distribution of the sum $R_i = \sum_{k=n-i+2}^{n} S_{ik}$, $i = 2,...,n$, need to be known. From the representation (II.3), it follows that

$$F_{R_i}(x) = (1-\theta) \cdot F_{R_i^{\perp}}(x) + \theta \cdot F_{R_i^{+}}(x), \quad i = 2,...,n , \tag{3.7}$$

where $R_i^{\perp} = \sum_{k=n-i+2}^{n} S^{\perp}_{ik}$ is the sum of the independent version $(S^{\perp}_{i1}, S^{\perp}_{i2}, ..., S^{\perp}_{in-i+1})$ of $S^{(i)}$ and $R_i^{+} = \sum_{k=n-i+2}^{n} S^{+}_{ik}$ is the sum of the comonotone version $(S^{+}_{i1}, S^{+}_{i2}, ..., S^{+}_{in-i+2})$ of $S^{(i)}$. Denote by

$Q_{ik}(u)$ the u-percentile value of S_{ik}. It is well-known that for a comonotone sum percentile values behave additively such that

$$F_{R_i^+}^{-1}(u) = \sum_{k=n-i+2}^{n} Q_{ik}(u), \quad u \in (0,1). \tag{3.8}$$

From this one obtains the identity

$$F_{R_i}\left(\sum_{k=n-i+2}^{n} Q_{ik}(u)\right) = (1-\theta) \cdot F_{R_i^+}\left(\sum_{k=n-i+2}^{n} Q_{ik}(u)\right) + \theta \cdot u, \quad i = 2,...,n. \tag{3.9}$$

Recall the assumption made at the end of Section 2 about the distribution $F_{R_i^+}(x)$, that is the approximation through a gamma distribution with parameters (2.13), and the fact that

$$Q_{ik}(u) = \Gamma^{-1}\left(u; \frac{V_i}{V}\alpha\right)\frac{1}{\alpha} x_i y_k, \tag{3.10}$$

where $\Gamma^{-1}(u;c)$ denotes the u-percentile of the "standardized" gamma distribution $\Gamma(c,1)$. Then, setting

$$x = \sum_{k=n-i+2}^{n} Q_{ik}(u) = \Gamma^{-1}\left(u; \frac{V_i}{V}\alpha\right)\frac{V}{V_i \alpha}\mu_i, \tag{3.11}$$

one notes that

$$u = \Gamma\left(\frac{V_i \alpha}{V \mu_i} x; \frac{V_i}{V}\alpha\right), \tag{3.12}$$

from which it follows that the distribution of the IBNR claims reserve in model (II) can be approximated by the analytical expression

$$F_{R_i}(x) = (1-\theta) \cdot \Gamma\left(\frac{1}{k_i^2 \mu_i} x; \frac{1}{k_i^2}\right) + \theta \cdot \Gamma\left(\frac{V_i \alpha}{V \mu_i} x; \frac{V_i}{V}\alpha\right), \quad \theta \in [0,1]. \tag{3.13}$$

Percentile values of the IBNR reserve at the confidence level u can now be determined numerically solving the equation $F_{R_i}(x) = u$. This can be done easily using any modern computer algebra system (MATHCAD, Maple, Mathematica, etc.). The expected value of the IBNR reserve, which is the quantity one is usually interested in, is obtained from the calculation:

$$\begin{aligned} E[R_i] &= \int_0^\infty [1 - F_{R_i}(x)]dx = (1-\theta) \cdot \int_0^\infty \left[1 - \Gamma\left(\tfrac{1}{k_i^2 \mu_i} x; \tfrac{1}{k_i^2}\right)\right]dx + \theta \cdot \int_0^\infty \left[1 - \Gamma\left(\tfrac{V_i \alpha}{V \mu_i} x; \tfrac{V_i \alpha}{V}\right)\right]dx \\ &= (1-\theta) \cdot \mu_i + \theta \cdot \mu_i = \mu_i. \end{aligned} \tag{3.14}$$

It coincides with the value obtained from model (I) with independent development years. To obtain the variance of the IBNR reserve, let us first evaluate the second order moment

$$E\left[R_i^2\right] = 2 \cdot \int_0^\infty x \cdot \left[1 - F_{R_i}(x)\right]dx = (1-\theta) \cdot 2 \cdot \int_0^\infty x \cdot \left[1 - \Gamma\left(\tfrac{1}{k_i^2 \mu_i} x; \tfrac{1}{k_i^2}\right)\right]dx$$

$$+ \theta \cdot 2 \cdot \int_0^\infty x \cdot \left[1 - \Gamma\left(\tfrac{V_i \alpha}{V \mu_i} x; \tfrac{V_i \alpha}{V}\right)\right]dx \tag{3.15}$$

$$= (1-\theta) \cdot \left(1 + k_i^2\right) \cdot \mu_i^2 + \theta \cdot \left(1 + \frac{V}{V_i \alpha}\right) \cdot \mu_i^2 = \mu_i^2 + \left[(1-\theta) \cdot k_i^2 + \theta \cdot \frac{V}{V_i \alpha}\right] \cdot \mu_i^2.$$

Therefore, the variances of the IBNR claims reserves equal

$$Var\left[R_i\right] = \left[(1-\theta) \cdot k_i^2 + \theta \cdot \frac{V}{V_i \alpha}\right] \cdot \mu_i^2. \tag{3.16}$$

In model (II), the two extreme cases are the independent case $\theta = 0$ and the comonotone case $\theta = 1$. In these situations, the percentile values are explicitly given by

$$Q_{R_i}^{\theta=0}(u) = \Gamma^{-1}\left(u; \frac{1}{k_i^2}\right) k_i^2 \mu_i, \tag{3.17}$$

$$Q_{R_i}^{\theta=1}(u) = \Gamma^{-1}\left(u; \frac{V_i}{V} \alpha\right) \frac{V}{V_i \alpha} \mu_i. \tag{3.18}$$

By construction of model (II), it is clear that

$$Q_{R_i}^{\theta=0}(u) \le Q_{R_i}^{\theta}(u) \le Q_{R_i}^{\theta=1}(u), \quad \theta \in (0,1), \tag{3.19}$$

and similarly

$$Var\left[R_i^{\theta=0}\right] = \left(k_i \mu_i\right)^2 \le Var\left[R_i^{\theta}\right] \le Var\left[R_i^{\theta=1}\right] = \left(\frac{V}{V_i \alpha}\right) \mu_i^2. \tag{3.20}$$

4. <u>A numerical example and a comparison</u>.

In the present Section, we illustrate the use of the introduced models and compare them with the results obtained through application of the recent optimal credible loss ratio IBNR reserving method considered in Hürlimann(2005). Recall first the design of the latter IBNR claims reserving method, which is inspired from and similar but different from the Benktander method reviewed in Mack(2000), Section 2.

Usually, the estimation of IBNR claims reserves is based on a full loss triangle of paid claims statistics, and some methods require additionally the knowledge of a measure of exposure for each underwriting period. The considered credible loss reserving method requires slightly less information. We suppose that there are n underwriting periods, for which one knows besides actuarial premiums V_i, $i = 1,...,n$, used as a measure of exposure, loss ratios m_k, $k = 1,...,n$, which represent the amount of claims per unit of actuarial premium required in the reporting period k. Additionally, we require the knowledge of the accumulated paid claims C_{in-i+1} for each underwriting period $i = 1,...,n$, which are reported in the latest period of development n.

Since the sum $\sum_{k=1}^{n} m_k$ represents the loss ratio over all reporting periods, the quantity $U_i^{BC} = V_i \cdot \sum_{k=1}^{n} m_k$ is nothing else than the loss ratio estimate or burning cost of the total ultimate claims required for the underwriting period i. This estimate is similar to the Bornhuetter/Ferguson prior estimate U_0 of the total ultimate claims in Mack(2000). By definition of the loss ratios, the quantities $V_i \cdot \sum_{k=1}^{n-i+1} m_k$, $i = 1,...,n$, represent the loss ratio estimate or burning cost of the paid claims for the underwriting period i, which are required in the current or latest period of development n. The *loss ratio payout* defined by

$$p_i = \frac{V_i \cdot \sum_{k=1}^{n-i+1} m_k}{U_i^{BC}} = \frac{\sum_{k=1}^{n-i+1} m_k}{\sum_{k=1}^{n} m_k}, \quad i = 1,...,n, \tag{4.1}$$

is a loss ratio estimate of the proportion of the total ultimate claims, which is expected to be paid for the underwriting period i. From this an estimate of the total ultimate claims is obtained by grossing up the latest accumulated paid claims amount. Since it is based solely on the individual latest claims experience of an underwriting year, it is called *individual total ultimate claims* amount and is given by

$$U_i^{ind} = \frac{C_{in-i+1}}{p_i}, \quad i = 1,...,n. \tag{4.2}$$

This estimate is similar to the chain-ladder estimate in Mack(2000). A corresponding estimate of the IBNR claims reserve, called *individual loss ratio IBNR claims reserve*, is defined by

$$R_i^{ind} = q_i \cdot U_i^{ind}, \quad i = 1,...,n, \tag{4.3}$$

where $q_i = 1 - p_i$ represents the proportion of the total ultimate claims, which is expected to be paid in the future for the underwriting period i. On the other side, the above burning cost estimate of the total ultimate claims leads to the alternative IBNR claims reserve

$$R_i^{coll} = q_i \cdot U_i^{BC}, \quad i = 1,...,n. \tag{4.4}$$

It is called *collective loss ratio IBNR claims reserve* because it depends solely on the portfolio claims experience of all underwriting periods.

Like the Bornhuetter/Ferguson and the chain-ladder estimates in Mack(2000), the considered collective and individual loss ratio IBNR claims reserve estimates represent extreme positions. Indeed, the individual claims reserve considers the latest accumulated claims amount to be fully credible predictive for future claims and ignores the prior burning cost estimate of the total ultimate claims, while the collective claims reserve ignores the current accumulated paid claims and relies fully on this prior estimate. Therefore, it is natural to apply the credibility mixture to those reserves and use the *credible loss ratio IBNR claims reserve* estimate

$$R_i^c = Z_i \cdot R_i^{ind} + (1 - Z_i) \cdot R_i^{coll}, \quad i = 1,...,n, \tag{4.5}$$

where Z_i is the credibility weight associated to the individual loss ratio reserve. To estimate the credibility weights, use of the special case $f = 1$ of the following more general result will be made.

Theorem 4.1. (Hürlimann(2005)) Under specific assumptions, the optimal credibility weights Z_i^* which minimize the mean squared error $mse(R_i^c) = E\left[(R_i^c - R_i)^2\right]$ and the variance $Var[R_i^c]$ are given by

$$Z_i^* = \frac{p_i}{p_i + t_i^*}, \text{ with } t_i^* = \frac{f - 1 + \sqrt{(f+1)(f - 1 + 2 \cdot p_i)}}{2}, \quad i = 1,...,n. \tag{4.6}$$

Our illustration is based on the published full loss triangle of paid claims and exposures in Mack(1997), Table 3.1.5.1:

Table 4.1: loss triangle of paid claims

underwriting period	development year 1	2	3	4	5	6
1	4'370	1'923	3'999	2'168	1'200	647
2	2'701	2'590	1'871	1'783	393	-
3	4'483	2'246	3'345	1'068	-	-
4	3'254	2'550	2'547	-	-	-
5	8'010	4'108	-	-	-	-
6	5'582	-	-	-	-	-

The following Tables 4.2 to 4.7 list the means, variances and percentile values of IBNR claims reserves for the gamma model (II) in the independent and comonotone cases ($\theta = 0$ and $\theta = 1$) as well as for the optimal credible method with minimum mean squared error and variance. The parameters of the models are estimated with the method of maximum likelihood.

Table 4.2: **Mean of IBNR reserves**

underwriting period	**mean of IBNR reserves :** optimal credible	gamma
2	628	416
3	1'637	1'186
4	3'133	2'651
5	8'246	10'959
6	12'004	11'826

Table 4.3: **Standard deviation of IBNR reserves**

	standard deviation of IBNR reserves :		
underwriting period	optimal credible	ind gamma	com gamma
2	48	87	87
3	121	167	234
4	245	351	539
5	561	1'263	2'109
6	1'024	1'072	2'081

Table 4.4: 80% **Percentile values of IBNR reserves**

	80% percentiles		
underwriting period	optimal credible	ind gamma	com gamma
2	668	487	487
3	1'738	1'324	1'377
4	3'337	2'940	3'090
5	8'714	12'006	12'683
6	12'857	12'717	13'532

Table 4.5: 90% **Percentile values of IBNR reserves**

	90% percentiles		
underwriting period	optimal credible	ind gamma	com gamma
2	691	531	531
3	1'794	1'405	1'494
4	3'451	3'109	3'361
5	8'972	12'606	13'734
6	13'334	13'218	14'559

Table 4.6: 95% **Percentile values of IBNR reserves**

	95% percentiles		
underwriting period	optimal credible	ind gamma	com gamma
2	709	569	569
3	1'841	1'474	1'595
4	3'547	3'252	3'595
5	9'190	13'116	14'644
6	13'737	13'642	15'444

Table 4.7: 99% **Percentile values of IBNR reserves**

	99% percentiles		
underwriting period	optimal credible	ind gamma	com gamma
2	746	645	645
3	1'932	1'610	1'796
4	3'731	3'533	4'063
5	9'606	14'110	16'453
6	14'515	14'461	17'198

One notes that the total IBNR claims reserves over all underwriting periods is equal to 27'037, which compares with 25'648 for the optimal credible method, which is a 5.4% additional charge. The differences between the IBNR reserves of the single underwriting periods are bigger and of changing signs. The same remark holds for the standard deviation and percentile values. The range of variation between the extreme independent and comonotone cases is important for the standard deviation of IBNR reserves but rather moderate for the percentiles at the confidence levels between 80% and 99%.

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